$$\sum_{k=1}^{n}\left(\frac{2(k+1)(k+2)^2}{((k+2)!)^3}\right)^{\frac{1}{k}} \geq \sum_{k=1}^{n}\frac{4}{(k+1)(k+2)(k+3)} \geq$$

$$\geq \sum_{k=1}^{n}\left(\frac{2}{(k+1)(k+2)}-\frac{2}{(k+2)(k+3)}\right) = \frac{2}{(1+1)(1+2)}-\frac{2}{(n+2)(n+3)} =$$

$$= \frac{1}{3}-\frac{2}{(n+2)(n+3)} = \frac{n(n+5)}{3(n+3)(n+2)}.$$

Arkady Alt

W35. (Solution by the proposer.) We have the followings:

$$1 \leq m^3+m^2-1 < m^3+m^3+m^3 = 3m^3 < m^5$$

for all $m \geq 2$, and

$$1 \leq m^2-m+1 < m^2+m^2+m^2 = 3m^2 < m^4$$

Therefore m^3+m^2-1 appear in product $(1\cdot 2\cdot \ldots \cdot m)^5$ and m^2-m+1 appear in product $(1\cdot 2\cdot \ldots \cdot m)^4$ therefore

$$m^5+m-1 = \left(m^3+m^2-1\right)\left(m^2-m+1\right)$$

appear in product $(1\cdot 2\cdot \ldots \cdot m)^9$. Therefore for $n = m^9$ we have infinitely many solutions.

W36. (Solution by the proposer.) First we will prove inequality

$$\triangle(a,b,c)\,\triangle\left(a^3,b^3,c^3\right) \leq 25F^4 = \triangle^2\left(a^2,b^2,c^2\right) \tag{1}$$

Due to homogeneity of inequality assume that semiperimeter s equal 1. Then , denoting $x := 1-a$, $y := 1-c$ and $p = xy+yz+zx$, $q := xyz$ we obtain $x+y+z = 1$, $x,y,z > 0$, $a = 1-x$, $b = 1-y$, $c = 1-z$ and

$$\sum_{cyc} a^3b^3 = \sum_{cyc}(x+yz)^3 = \sum_{cyc}\left(x^3+3x^2yz+3xy^2z^2+x^3y^3\right) =$$

$$1+3q-3p+3q+3pq+p^3+3q^2-3pq$$

$$\sum_{cyc} a^3 = \sum_{cyc} (1-x)^3 = 3 - 3 + 3(1-2p) - (1+3q-3p) = 2 - 3q - 3p$$

Then

$$F = \sqrt{q}, \triangle(a,b,c) = 4p, \triangle(a^2,b^2,c^2) =$$

$$= 16F^2, \triangle(a^3,b^3,c^3) = 36q - 18pq - 9p^2 + 4p^3 + 3q^2$$

and

$$h(p,q) := \triangle^2(a^2,b^2,c^2) - \triangle(a,b,c)\triangle(a^3,b^3,c^3) =$$

$$= 256q^2 - 4p(36q - 18pq - 9p^2 + 4p^3 + 3q^2) =$$

$$= -16p^4 + 36p^3 + (256-12p)q^2 - 72p(2-p)q$$

Recall that system of equations

$$\begin{cases} x+y+z=1 \\ xy+yz+zx=p \\ xyz=q \end{cases}$$

have solutions is nonnegative x,y,z iff $p,q \ge 0$ and

$$27q^2 - 2(9p-2)q + 4p^3 - p^2 \le 0 \tag{2}$$

Since (2) yields $1-3p \ge 0$ then denoting $t := \sqrt{1-3p}$ we obtain $p = \frac{1-t^2}{3}$ and for (2) effective equivalent form

$$\max\{0, q_*\} \le q \le q^* \tag{3}$$

where $q_* := \frac{(1+t)^2(1-2t)}{27}$, $q^* := \frac{(1-t)^2(1+2t)}{27}$ and $0 \le t \le 1 \Leftrightarrow 1-3p \ge 0$. Also note, that $(256-12p)q^2 - 72p(2-p)q$ is decreasing in $q \le \frac{p}{9} = \frac{1-t^2}{27}$ since $256-12p > 0$ and

$$\frac{72p(2-p)}{2(256-12p)} > \frac{p}{9} \Leftrightarrow 36\cdot 9(2-p) - 256 + 12p = 392 - 312p = 8(49-39p) > 0$$

Hence

$$h(p,q)=h\left(\frac{1-t^2}{3},q\right)\geq h\left(\frac{1-t^2}{3},q^*\right)=$$

$$=-16\left(\frac{1-t^2}{3}\right)^4+36\left(\frac{1-t^2}{3}\right)^3+\left(256-12\cdot\left(\frac{1-t^2}{3}\right)\right)\cdot$$

$$\cdot\left(\frac{(1-t)^2(1+2t)}{27}\right)^2-72\left(\frac{1-t^2}{3}\right)\left(2-\left(\frac{1-t^2}{3}\right)\right)\cdot\frac{(1-t)^2(1+2t)}{27}=$$

$$=\frac{128}{729}t^2(t+2)(1-t)^5\geq 0$$

Using (1) and Hadwiger - Finsler inequality in $p-q$ notation

$$\triangle(a,b,c)\geq 4\sqrt{3}F\Leftrightarrow\triangle^2(a,b,c)\geq 3\triangle\left(a^2,b^2,c^2\right)\Leftrightarrow p^2\geq 3q$$

we obtain

$$\triangle^4\left(a^2,b^2,c^2\right)\geq\triangle^2(a,b,c)\cdot\triangle^2\left(a^3,b^3,c^3\right)\geq 3\triangle\left(a^2,b^2,c^2\right)\cdot\triangle^2\left(a^3,b^3,c^3\right)\Rightarrow$$

$$\Rightarrow\triangle^3\left(a^2,b^2,c^2\right)\geq 3\triangle^2\left(a^3,b^3,c^3\right)\Leftrightarrow(16q)^3\geq 3\triangle^2\left(a^3,b^3,c^3\right)\Leftrightarrow$$

$$\Leftrightarrow(4q)^3\geq\sqrt{3}\triangle\left(a^3,b^3,c^3\right)\Leftrightarrow\frac{64F^3}{\sqrt{3}}\geq\triangle\left(a^3,b^3,c^3\right)\Leftrightarrow\left(4\sqrt{3}F\right)^3\geq$$

$$\geq 27\frac{\triangle\left(a^3,b^3,c^3\right)}{3}\Leftrightarrow$$

$$\Leftrightarrow 4\sqrt{3}F\geq 3\sqrt[3]{\frac{\triangle\left(a^3,b^3,c^3\right)}{3}}\Rightarrow\triangle(a,b,c)\geq 4\sqrt{3}F\geq 3\sqrt[3]{\frac{\triangle\left(a^3,b^3,c^3\right)}{3}}$$

Remark. *Inequality*

$$\triangle\left(a^3,b^3,c^3\right)\leq\frac{64F^3}{\sqrt{3}}$$

is another form of the following Power $\triangle$-Mean Inequality

$$\sqrt{\frac{\triangle(a^2,b^2,c^2)}{3}} \geq \sqrt[3]{\frac{\triangle(a^3,b^3,c^3)}{3}}$$

for a,b,c sidelenghs of a triangle, or equivalently for positive a,b,c such that

$$\triangle\left(a^2,b^2,c^2\right) \geq 0$$

Indeed, if

$$\triangle\left(a^2,b^2,c^2\right) \geq 0 \Leftrightarrow a+b \geq c.b+c \geq a, c+a \geq b,$$

then

$$\triangle\left(a^2,b^2,c^2\right) = 16F^2$$

where F is area of triangle (possible degenerated) defined by a,b,c as sidelenghs. Then

$$\sqrt{\frac{\triangle(a^2,b^2,c^2)}{3}} \geq \sqrt[3]{\frac{\triangle(a^3,b^3,c^3)}{3}} \Leftrightarrow \sqrt{\frac{16F^2}{3}} \geq \sqrt[3]{\frac{\triangle(a^3,b^3,c^3)}{3}} \Leftrightarrow$$

$$\Leftrightarrow \frac{4F}{\sqrt{3}} \geq \sqrt[3]{\frac{\triangle(a^3,b^3,c^3)}{3}} \Leftrightarrow 64F^3 \geq \sqrt{3}\,\triangle\left(a^3,b^3,c^3\right)$$

Second solution. Using Ravi and setting, we get $a = y+z$, $b = z+x$, $c = x+y$, $p = x+y+z$, $q = xy+yz+zx$, $r = xyz$

$$\triangle(a,b,c) = 2(xy+yz+zx) - x^2 - y^2 - z^2 = 2\sum(y+z)(z+x) - \sum(x+y)^2$$

$$\triangle(a,b,c) = 4(xy+yz+zx) = 4q \qquad (1)$$

$$16F^2 = \triangle\left(a^2,b^2,c^2\right) = 2\sum a^2b^2 - \sum a^4$$

$$\triangle\left(a^2,b^2,c^2\right) = 2\sum\left[(y+z)(z+x)\right]^2 - \sum(x+y)^4 = 16pr \qquad (2)$$

$$\triangle\left(a^3,b^3,c^3\right) = 2\sum a^3b^3 - \sum a^6 = 2\sum\left[(y+z)(z+x)\right]^3 - \sum(x+y)^6$$

$$\triangle\left(a^3,b^3,c^3\right) = 6q^3 + 6q^2\sum x^2 + 6q\sum x^4 + 2\sum x^6 - 2\sum x^6 - 6\sum xy\left(x^4+y^4\right) -$$

$$-15\sum x^2y^2\left(x^2+y^2\right)-20\sum x^3y^3\left(x^2+y^2\right)$$

Using the identities

$$\sum xy\left(x^4+y^4\right)=-3r^2+7pqr-p^3r+p^4q-4p^2q^2+2q^3$$

$$\sum x^3y^3\left(x^2+y^2\right)=-3r^2-2p^3r+4pqr+p^2q^2-2q^3$$

$$\sum a^3b^3=q^3-3pqr+3r^2$$

So,

$$\triangle\left(a^3,b^3,c^3\right)=14q^3-14p^2q^2-18pqr+3r^2+36p^3r \tag{3}$$

Let function

$$f\left(q\right)=14q^3-14p^2q^2-18pqr+3r^2+36p^3r-\frac{64pr\sqrt{pr}}{\sqrt{3}}$$

We have

$$f'\left(q\right)=42q^2-28p^2q-18pr$$

$$f''\left(q\right)=84q-28p^2=28\left(3q-p^2\right)\leq 0$$

because $p^2\geq 3q$. So $f\left(q\right)\leq 0$.

Nicuşor Zlota

W37. (Solution by the proposer.) a). First recall some properties of P. By condition $x-P\left(x\right)\perp F$ for any $x\in E$, there is $\left(x-P\left(x\right)\right)\cdot y=0$ for any $y\in F$. Then

1. For any $x\in F$ we have $P\left(x\right)=x$ because $P\left(x\right)\in F$,

$$x-P\left(x\right)\in F\Rightarrow\left(x-P\left(x\right)\right)\cdot\left(x-P\left(x\right)\right)=0\Leftrightarrow P\left(x\right)=x$$

2. $x\cdot P\left(y\right)=y\cdot P\left(x\right)$, for any $x,y\in E$. Indedd

$$\left(x-P\left(x\right)\right)\cdot\left(y-P\left(y\right)\right)=\left(x-p\left(x\right)\right)\cdot y-\left(x-P\left(x\right)\right)\cdot P\left(y\right)=x\cdot y-P\left(x\right)\cdot y$$

and

$$(x - P(x)) \cdot (y - P(y)) = x \cdot (y - P(y)) - P(x) \cdot (y - P(y)) =$$

$$= x \cdot y - x \cdot P(y) \Rightarrow x \cdot y - P(x) \cdot y = x \cdot y - x \cdot P(y) \Leftrightarrow x \cdot P(y) = y \cdot P(x)$$

3. $\|x\|^2 = \|P(x)\|^2 + \|x - P(x)\|^2$, for any $x \in E$.

$$\|P(x)\|^2 + \|x - P(x)\|^2 = \|P(x)\|^2 + (x - P(x)) \cdot (x - P(x)) =$$

$$= \|x\|^2 - 2x \cdot P(x) + 2P(x) \cdot P(x) = \|x\|^2 - 2(x - P(x)) = \|x\|^2$$

4. For any $x \in E$, holds $\|x - 2P(x)\| = \|x\|$. Indeed

$$\|x - 2P(x)\|^2 = (x - 2P(x)) \cdot (x - 2P(x)) = \|x - P(x)\|^2 - 2P(x) \cdot (x - P(x)) +$$

$$+ \|P(x)\|^2 = \|P(x)\|^2 + \|x - P(x)\|^2 = \|x\|^2$$

Using property 4 and Cauchy inequality we get:

$$\|x\| \|y\| = \|x\| \|y - 2P(y)\| \geq x \cdot (y - 2P(y))$$

and

$$\|x\| \|y\| = \|x - 2P(x)\| \|y\| \geq (x - 2P(x)) \cdot y$$

Adding this inequalities we obtain

$$2 \|x\| \|y\| \geq x \cdot (y - 2P(y)) + (x - 2P(x)) \cdot y \Leftrightarrow$$

$$\|x\| \|y\| \geq x \cdot y - x \cdot P(y) - y \cdot P(x) \tag{1}$$

By substitution $y := -y$ in inequality (1) we obtain inequality

$$\|x\| \|y\| \geq -x \cdot y + x \cdot P(y) + y \cdot P(x) \Leftrightarrow$$

$$\Leftrightarrow x \cdot y - x \cdot P(y) - y \cdot P(x) \geq - \|x\| \|y\| \tag{2}$$

Thus, for any $x, y \in E$ holds inequality

$$-\|x\|\,\|y\| \le x\cdot y - x\cdot P(y) - y\cdot P(x) \le \|x\|\,\|y\| \Leftrightarrow$$

$$\Leftrightarrow |x\cdot y - x\cdot P(y) - y\cdot P(x)| \le \|x\|\,\|y\| \tag{3}$$

b). By condition of equality in inequality Cauchy equality in (1) equality occurs iff

$$y - 2P(y) = kx,\ x - 2P(x) = lx$$

where $k, l > 0$.
From identities $\|y - 2P(y)\| = \|y\|, \|x - 2P(x)\| = \|x\|$ follows $k = \frac{\|y\|}{\|x\|}$ and $l = \frac{\|x\|}{\|y\|}$.
Thus we have

$$y - 2P(y) = \frac{x\,\|y\|}{\|x\|} \Leftrightarrow y\,\|x\| - x\,\|y\| = 2P(y)\,\|x\| \tag{4}$$

and

$$x - 2P(x) = \frac{y\,\|x\|}{\|y\|} \Leftrightarrow x\,\|y\| - y\,\|x\| = 2P(x)\,\|y\| \tag{5}$$

Adding and substracting (4) and (5) we obtain:

$$0 = P(x)\,\|y\| + P(y)\,\|x\| = P(x\,\|y\| + y\,\|x\|) \Leftrightarrow x\,\|y\| + y\,\|x\| \in F^1 \tag{6}$$

and

$$P(x)\,\|y\| - P(y)\,\|x\| = \|y\| - y\,\|x\| \Leftrightarrow P(x\,\|y\| - y\,\|x\|) = x\,\|y\| - y\,\|x\| \Leftrightarrow$$

$$\Leftrightarrow x\,\|y\| - y\,\|x\| \in F \tag{7}$$

It is mean there are $h \in F^1$ and $f \in F$ that

$$\frac{x}{\|x\|} + \frac{y}{\|y\|} = 2h \text{ and } \frac{x}{\|x\|} - \frac{y}{\|y\|} = 2f \Leftrightarrow \frac{x}{\|x\|} = h + f \text{ and } \frac{y}{\|y\|} = h - f$$

Thus $\|h\|^2 + \|f\|^2 = 1$ and $x = a(h+f)$, $y = b(h-f)$, where a, b be arbitrary positive real numbers, and we have

$$x \cdot y - x \cdot P(y) - y \cdot P(x) =$$

$$= ab\left(\|h\|^2 - \|f\|^2\right) - ab(h+f) \cdot P(h-f) - ab(h-f) \cdot P(h+f) =$$

$$= ab\left(\|h\|^2 - \|f\|^2 + \|f\|^2 + \|f\|^2\right) = ab\left(\|h\|^2 + \|f\|^2\right) = ab = \|x\|\,\|y\|$$

According to substitution $y := -y$ which transform inequality (1) to inequality (2) we obtain equality condition in inequality (2) : $x = a(h+f)$, $y = b(f-h)$, where a, b be arbitrary positive real numbers and $h \in F^1$, $f \in F$ with $\|h\|^2 + \|f\|^2 = 1$.
So, in inequality (3) equality occurs if $x = a(h+f)$, $y = b(h-f)$, where a, b be arbitrary real numbers and $h \in F^1$, $f \in F$ with $\|h\|^2 + \|f\|^2 = 1$.
Comment. Since $x \cdot P(y) = y \cdot P(x)$ inequality (3) can be rewritten in assymetric form

$$|x \cdot y - 2x \cdot P(y)| \leq \|x\|\,\|y\| \text{ or } |x \cdot y - 2y \cdot P(x)| \leq \|x\|\,\|y\|$$

W38. (Solution by the proposer.) We have the following:

$$\frac{4^x + 2^x + 1}{3} \geq \sqrt[3]{8^x} = 2^x > x$$

for $x \in R$, so

$$4^x + 2^x + 1 > 3x, \quad \frac{4^x + 2^x + 1}{x} > 3 > 1$$

for all $x > 0$ and

$$\left(\frac{4^x + 2^x + 1}{x}\right)^x > 3^x > 2^x \text{ so } \left(\frac{4^x + 2^x + 1}{x}\right)^x - 2^x > 0$$

for all $x > 0$. We prove that

$$(1 + 2^x + 4^x) < x^x (1 + 2^x)$$

for all $x \in \left(0, \frac{1}{2e}\right)$. We take $f(x) = (1 + 2^x + 4^x)^x$ and $g(x) = x^x(1 + 2^x)$, when f is strivtly increasing function,